- **Induction** is a logical reasoning method that starts from specific cases or instances and generalizes to universal laws or principles. A **conjecture** is an unproven hypothesis or theory proposed based on existing knowledge and intuition.
- **Induction and conjecture** often interact in the research process. Induction may lead to the formation of new conjectures, while conjectures can potentially gain support through induction from experimental or observational data.

For example, observation

$1=1^2$

$1+3=2^2$

$1+3+5=3^2$

$1+3+5+7=4^2$

… …

Conjecture: The sum of the first n odd numbers equals n^2,

$$1+3+5+\ldots+(2n-1)=n^2$$

- Proposition **form**: $\forall x(x \in N \wedge x \geq n_0)$, P(x)
- **Base case**: Prove $P(n_0)$ is true
- Inductive **step**: For all $x(x \geq n_0)$, assume P(x) is true, prove P(x+1) is true.
  The statement "P(x) is true" is called the **induction hypothesis**.

e.g. **Example 8:** Proof: For all $n \geq 1$, $1+3+5+\ldots+(2n-1)=n^2$

Proof:

Base case: When n=1, $1=1^2$, the conclusion holds.

Inductive step: Assume the conclusion holds for $n \geq 1$, then

$1+3+5+\ldots+(2n-1)+(2n+1)=n^2+(2n+1) = (n+1)^2$

Thus, the conclusion also holds when n+1.

- Note: Both the base case and the inductive step are essential.

e.g. »» Example(1) Proposition: For all n≥1, $2^1+2^2+ \ldots + 2^n= 2^{n+1} - 2$

Proof:

Base case: When n=1, $2^1 = 2^{1+1} - 2=2$.

Inductive hypothesis: Assume the statement holds for some arbitrary n≥1. Then,

$2^1+2^2+ \ldots + 2^n+2^{n+1}= 2^{n+1}+2^{n+1} -2= 2^{n+2} - 2$.

Thus, the proposition holds for n+1.

↳ Proof by Counterexample

- The method of **counterexamples** involves finding one or more examples to refute a universal statement. This method is typically used to prove that a proposition is incorrect.

e.g. **Example(2):** Observe whether $\mathbf{2^{n-1}-1}$ is divisible by n.

n	2^{n-1}-1	divisible	N	2^{n-1}-1	divisible
3	3	Y	10	511	N
4	7	N	11	1023	Y
5	15	Y	12	2047	N
6	31	N	13	4095	Y
7	63	Y	14	8191	N
8	127	N	15	16383	N
9	255	N	16	32767	N

1.3 Proof Methods

↳ Proof by Counterexample

n	2^{n-1}-1	整除	n	2^{n-1}-1	整除
17	65535	Y	21	1048575	N
18	131071	N	22	2097151	N
19	262143	Y	23	4194303	Y
20	524287	N	24	8388607	N

- From the table, we might induce the following proposition:

Let n≥3. A necessary and sufficient condition for n to be a prime number is that $2^{n-1}-1$ is divisible by n. However, this proposition is not true.

Counterexample: 561=3×11×17 is a composite number, yet 2560−1 is divisible by 561.

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- **Complete Induction** (also known as **Strong Induction**) is a proof technique used to establish the truth of a statement for all integers greater than or equal to a certain number. It is an extension of the principle of mathematical induction, where, instead of assuming the truth of the statement only for the immediate predecessor (i.e., P(k)), it assumes the truth of the statement for all values from the base case up to k.
- **Base Case:** Prove that $P(n_0)$ is true.
- **Inductive Step:** For all x (where $x \geq n_0$), **assume** $P(n_0), P(n_0+1), \ldots, P(x)$ are true, and prove that $P(x+1)$ is true.
- **Inductive Hypothesis:** For all y (where $n_0 \leq y \leq x$), $P(y)$ is true.

e.g. **Example 9:** Every integer greater than or equal to 2 can be expressed as a product of prime numbers.

Proof:

Base Case: For n=2, the statement is obviously true.

Inductive Step: Assume that for all integers k such that 2≤k≤n, the statement holds true. We need to show that the statement also holds for n+1.

- If n+1 is a prime number, then the statement is true.
- If n+1 is not a prime number, then it can be expressed as n+1=a×b, where 2≤a,b<n.
- By the inductive hypothesis, both a and b can be expressed as products of prime numbers. Therefore, n+1 can also be expressed as a product of prime numbers. **Thus**, the statement holds for n+1.

The base case of strong induction does not merely assume that the proposition holds for a specific natural number n; instead, it assumes that the proposition holds for all natural numbers less than or equal to n. Based on this stronger assumption, we then prove that the proposition holds for n + 1.

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e.g. **Example 10:** Any postal fee of n cents (where n≥12) can be composed using 4-cent and 5-cent stamps.

Proof:

Base Case: For n=12,13,14,15, the following combinations are possible:

12=3×4,
13=2×4+5,
14=2×5+4,
15=3×5.

> ⓘ The base case of strong induction does not merely assume that the proposition holds for a specific natural number n; instead, it assumes that the proposition holds for all natural numbers less than or equal to n. Based on this stronger assumption, we then prove that the proposition holds for n + 1.

Thus, the conclusion holds for n=12,13,14,15.

Inductive Step: Let n≥15, and assume that the conclusion holds for all integers from 12 to n.

- Consider n+1. Since n−3≥12 and by the inductive hypothesis, n−3 cents can be composed using 4-cent and 5-cent stamps.
- By adding one 4-cent stamp to the composition of n−3 cents, we obtain n+1 cents.

Therefore, the conclusion holds for n+1.

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- The **Proof by Counterexample** is a logical proof technique used to demonstrate that a general statement or conjecture is false by providing a specific example that contradicts the statement

e.g. **Example 11:** Prove that the following statement does not hold:

If $A \cap B = A \cap C$, then $B = C$.

Proof:

- ***Counterexample:*** Let A={a,b}, B={a,b,c}, and C={a,b,d}.
- Then, $A \cap B = \{a,b\}$, $A \cap C = \{a,b\}$. Thus, $A \cap B = A \cap C$.
- However, $B \neq C$ because $c \notin C$ and $d \notin B$.
- **Therefore**, the statement does not hold

1.4 Recursive Definition (Inductive Definition)

- **Recursive Definition** is a process where a concept is defined in terms of itself.
- **Example:** The sequence an can be recursively defined as follows:

$a^0=1$, $a^n = a^{n-1} \cdot a$, for n=1,2,…

e.g. **Example 12:** The Fibonacci sequence $\{f_n\}$ is recursively defined as follows:

- $f_0=1$, $f_1=1$, $f_n=f_{n-1}+f_{n-2}$, for n=2,3,…Calculating the first few terms:

$f_0=1$, $f_1=1$, $f_2=2$, $f_3=3$, $f_4=5$, $f_5=8$, $f_6=13$

e.g. **Example 13:** The set A is recursively defined as follows:

- **Base Step:** $3 \in A$
- **Recursive Step(Rule):**

  (1) If $x, y \in A$, then $x + y \in A$.

  (2) All numbers obtained by applying the recursive step a finite number of times belong to the set A.

$$A=\{3n \mid n \in Z^{+}\}$$

e.g. »»**Example 14:** Arithmetic expressions are defined inductively as follows:

(1) Any real **number and variable** is an arithmetic expression.

(2) If ***f*** and ***g*** are arithmetic expressions, then **(*f* + *g*), (*f*−*g*),** and (***f***×***g***) are also arithmetic expressions.

(3) If ***f*** and ***g*** are arithmetic expressions and ***g***≠0, then (***f*/*g***) is an arithmetic expression.

(4) If ***f*** is an arithmetic expression, then for all $n \in \mathbf{Z}^+$, f^n (or ***f***↑***n***) is an arithmetic expression.

(5) Only those expressions obtained by a finite number of applications of rules (1) through (4) are considered arithmetic expressions.

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Objective :

Key Concepts ：